Phase transition in an asymmetric generalization of the zero-temperature q-state Potts model

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An asymmetric generalization of the zero-temperature q-state Potts model on a one-dimensional lattice, with and without boundaries, has been studied. The dynamics of the particle number, and especially the large time behavior of the system, has been analyzed. In the thermodynamic limit, the system exhibits two kinds of phase transitions, a static and a dynamic phase transition.

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I. INTRODUCTION

In recent years, reaction-diffusion systems have been studied by many people. As mean-field techniques, generally, do not give correct results for low-dimensional systems, people are motivated to study exactly solvable stochastic models in low dimensions. Moreover, solving one-dimensional systems should in principle be easier. Exact results for some models on a one-dimensional lattice have been obtained, for example in [1–12], and in [13,14] for the case of multispecies systems. Different methods have been used to study these models, including analytical and asymptotic methods, mean-field methods, and large-scale numerical methods.

Some interesting problems in nonequilibrium systems are nonequilibrium phase transitions described by phenomenological rate equations, and the way the system relaxes to its steady state. Kinetic generalizations of the Ising model, for example the Glauber model or the Kawasaki model, are such phenomenological models and have been studied extensively [15–19]. Combination of the Glauber and the Kawasaki dynamics has been also considered [20–22]. One of the models that has been extensively studied is the q-state Potts model evolving according to some generalization of the zero-temperature Glauber dynamics (see [23–28], for example). The evolution tends to align all the spins, and so domains of parallel spins grow with time.

In this paper, we want to study an asymmetric generalization of the zero-temperature q-state Potts model on an infinite lattice with and without boundaries. There are also reactions at the boundaries. The dynamics of the particle number, and specially the large time behavior of the system is studied. In the thermodynamic limit, the system shows two kinds of phase transitions. One of these is a static phase transition, the other a dynamic one. The static phase transition is controlled by the reaction rates, and is a discontinuous change of the behavior of the derivative of the stationary particle density at the end points, with respect to the reaction rates. The dynamic phase transition is controlled by the reaction rates at boundaries, and is a discontinuous change of the relaxation time towards the stationary configuration. There is a *fast phase* and a *slow phase*. Increasing the reaction rates at the boundaries, the system goes from the *slow phase* to the *fast phase*. This model may be considered as a biased voting model, in the sense that there are q different opinions, and when people at the adjacent sites have different opinions, they may interact so that their opinions become the same.

II. ASYMMETRIC q-STATE POTTS MODEL AT ZERO TEMPERATURE

In the ordinary Glauber model, the interaction is between three neighboring sites. Spin flip brings the system to equilibrium with a heat bath at temperature T. A spin is flipped with the rate $\mu = 1 - \tanh[J/(kT)]$, if the spin of both of its neighboring sites are the same as itself; and is flipped with the rate $\lambda = 1 + \tanh[J/(kT)]$, if the spin of both of its neighboring sites are opposite to it. At domain boundaries, the spins are flipped with unit rate. So the interactions can be written as,

$$\begin{array}{lllll} AAA \to A\varnothing A & \text{and} & \varnothing\varnothing\varnothing \to \varnothing A\varnothing & \mu, \\ A\varnothing A \to AAA & \text{and} & \varnothing A\varnothing \to \varnothing\varnothing\varnothing & \lambda, \\ AA\varnothing \rightleftharpoons A\varnothing\varnothing & \text{and} & \varnothing\varnothing A \rightleftharpoons \varnothing AA & 1, \end{array}$$

where spin up and spin down are denoted by A and $\varnothing$. One can interpret an up spin as a particle, and a down spin as a hole. At zero temperature, the Glauber dynamics is effectively a two-site interaction [27]

$$A\varnothing \to (AA, \varnothing\varnothing) \quad \varnothing A \to (AA, \varnothing\varnothing) \tag{1}$$

where all the above processes occur with the same rate.

One can consider the following interactions, as an asymmetric generalization of the zero-temperature Glauber model,

$$A\varnothing \to \begin{cases} AA, & u \\ \varnothing\varnothing, & v \end{cases} \tag{2}$$

and

$$\varnothing A \to \begin{cases} AA, & v \\ \varnothing\varnothing, & u. \end{cases} \tag{3}$$

If $u \neq v$, the above system has left-right asymmetry. The above system on an infinite lattice has been investigated in

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[29], where its *n*-point functions, its equilibrium states, and its relaxation towards these states are studied. It can be easily shown that the time evolution equation for the average particle numbers of the system with the above interactions are the same as that of a system with the following interactions, where diffusion is also present:

$$A\varnothing \rightarrow \begin{cases} \varnothing A, & \lambda \\ AA, & u-\lambda \\ \varnothing\varnothing, & v-\lambda \end{cases} \tag{4}$$

and

$$\varnothing A \rightarrow \begin{cases} A\varnothing, & \mu \\ AA, & v-\mu \\ \varnothing\varnothing, & u-\mu. \end{cases} \tag{5}$$

The model addressed in this article is an asymmetric generalization of the zero-temperature *q*-state Potts model in one dimension and with Glauber dynamics. The interaction is a nearest neighbor interaction and is defined as follows:

$$A_\alpha A_\beta \rightarrow \begin{cases} A_\alpha A_\alpha, & u \\ A_\beta A_\beta, & v. \end{cases} \tag{6}$$

Similar to Eqs. (4) and (5), adding diffusion do not alter the time evolution equation of the average particle numbers. The symmetric case ($u=v$) has been recently analyzed using a generalization of the empty interval method [28]. The one-species case ($q=2$) has been studied in [29] (without boundaries) and [30] (with boundaries). It is shown that for the infinite lattice without boundaries, the system has two ground state. The sites are all occupied or all vacant. This is reminiscent of the ordinary Glauber model, and asymmetry does not change the ground states. Denoting by $|0\rangle$ ($|\Omega\rangle$) the empty (full) lattice, the state of the system at infinitely large times is

$$|P(\infty)\rangle = (1-\rho_0)|0\rangle + \rho_0|\Omega\rangle, \tag{7}$$

where ρ_0 is the initial average density. For the asymmetric generalization of the *q*-state Potts model, there are *q* ground states. To see this, one can divide the state at each site to two classes, A_1 and not-A_1, for example. The state not-A_1 is denoted by B_1. So the interactions are

$$A_1 B_1 \rightarrow \begin{cases} A_1 A_1, & u \\ B_1 B_1, & v \end{cases} \tag{8}$$

and

$$B_1 A_1 \rightarrow \begin{cases} B_1 B_1, & u \\ A_1 A_1, & v. \end{cases} \tag{9}$$

Similar to the asymmetric zero-temperature Glauber model, the ground states are the states such that all the sites of the lattice are in the state A_1, or none of the sites are in the state A_1. Repeating the same argument for the substates of B_1, it is seen that there are *q* ground states. Each ground state is a state in which the states of all lattice sites are of the same state, A_α for example. So the final state is

$$|P(\infty)\rangle = \sum_\alpha \rho_0^\alpha |A_\alpha\rangle, \tag{10}$$

where $|A_\alpha\rangle$ is the state for which all the sites are in the state A_α, and ρ_0^α is the initial average density of the state A_α. Then, all the correlation functions at infinitely large times can be obtained easily

$$\langle n_i^\alpha n_j^\beta \cdots n_k^\gamma \rangle = \delta^{\alpha\beta}\delta^{\alpha\gamma}\cdots\rho_0^\alpha. \tag{11}$$

The evolution equation for the average number of the state α at the site *j*, $\langle n_j^\alpha \rangle$ is

$$\langle \dot{n}_j^\alpha \rangle = -(u+v)\langle n_j^\alpha \rangle + u\langle n_{j-1}^\alpha \rangle + v\langle n_{j+1}^\alpha \rangle. \tag{12}$$

The above equation shows that the evolution equations for the average number of different states are decoupled, and so the average number of each state depends only on the initial value of the average number of that state. In fact,

$$\langle n_j^\alpha(t) \rangle = e^{-(u+v)t} \sum_m \left(\frac{u}{v}\right)^{(m-j)/2} I_{m-j}(2\sqrt{uv}\,t)\langle n_j^\alpha(0)\rangle, \tag{13}$$

and at large times,

$$\langle n_j^\alpha(t)\rangle - \rho_0^\alpha \sim \left(\frac{u}{v}\right)^{j/2} e^{[-(u+v)+2\sqrt{uv}]/\sqrt{t}.} \tag{14}$$

It is seen from the above equation that if $u<v$, the expectation at the rightmost sites tend rapidly to their final value, and obviously for $u>v$ the leftmost sites arrive earlier to their final states.

III. *q*-STATE POTTS MODEL ON A LATTICE WITH THE BOUNDARIES

In this section, we study *q*-state Potts model on a lattice with reaction at the boundaries. The interactions on the bulk of lattice are Eq. (6). The exchange of the states at the first site is

$$A_\beta \rightarrow A_\alpha \quad \text{with the rate} \quad \Lambda_\beta^\alpha, \tag{15}$$

and at the final site, it is

$$A_\beta \rightarrow A_\alpha \quad \text{with the rate} \quad \Gamma_\beta^\alpha. \tag{16}$$

For $\alpha \neq \beta$, Λ_β^α and Γ_β^α are rates and should be non-negative. The diagonal elements of Λ and Γ are chosen so that

$$s_\alpha \Lambda_\beta^\alpha = s_\alpha \Gamma_\beta^\alpha = 0, \tag{17}$$

where

$$s_\alpha = 1. \tag{18}$$

From now on, a repeated subscript and superscript imply a summation over the repeated index.

The equations of motion for the average numbers are

$$\langle \dot{n}_j^\alpha \rangle = -(u+v)\langle n_j^\alpha \rangle + u\langle n_{j-1}^\alpha \rangle + v\langle n_{j+1}^\alpha \rangle, \quad j \neq 1,L$$

$$\langle \dot{n}_1^\alpha \rangle = \Lambda_\beta^\alpha \langle n_1^\beta \rangle - v\langle n_1^\alpha \rangle + v\langle n_2^\alpha \rangle, \tag{19}$$

$$\langle \dot{n}_L^\alpha \rangle = \Gamma_\beta^\alpha \langle n_L^\beta \rangle - u\langle n_L^\alpha \rangle + u\langle n_{L-1}^\alpha \rangle.$$

Defining the vector N_k through

$$N_k^\alpha = \langle n_k^\alpha \rangle, \tag{20}$$

Eq. (19) takes the form

$$\dot{N}_k = -(u+v)N_k + uN_{k-1} + vN_{k+1}, \quad k \neq 1,L$$

$$\dot{N}_1 = \Lambda N_1 - vN_1 + vN_2, \tag{21}$$

$$\dot{N}_L = \Gamma N_L - uN_L + uN_{L-1}.$$

For the stationary state,

$$\dot{N}_k = 0. \tag{22}$$

Substituting the ansatz

$$N_k = Bz_1^k + Cz_2^k, \tag{23}$$

in the first equation of Eq. (21), one arrives at

$$-(u+v) + vz_i + \frac{u}{z_i} = 0, \tag{24}$$

whose solutions are $z_1 = 1$, $z_2 = u/v$. First assume $u < v$. Then, one can write Eq. (23) as

$$N_k = B + C'\left(\frac{u}{v}\right)^{k-1}. \tag{25}$$

Since $s_\alpha N_k^\alpha = 1$, it is seen that, B and C' satisfy

$$s_\alpha B^\alpha = 1,$$

$$s_\alpha C'^\alpha = 0. \tag{26}$$

Using the second and third equations of Eq. (21), one arrives at

$$(\Lambda - v)[B + C'] + v\left[B + C'\left(\frac{u}{v}\right)\right] = 0,$$

$$(\Gamma - u)\left[B + C'\left(\frac{u}{v}\right)^{L-1}\right] + u\left[B + C'\left(\frac{u}{v}\right)^{L-2}\right] = 0. \tag{27}$$

In the thermodynamic limit ($L \to \infty$), these two equations yield

$$\Gamma B = 0,$$

$$(\Lambda - v + u)C' = -\Lambda B. \tag{28}$$

The first equation in Eq. (28) can be used to determine B. Substituting it in the second one, C' is obtained. The degeneracy of the zero eigenvalue of Γ is equal to the degeneracy of the stationary average particle number. The matrix Γ does have a zero eigenvalue, since s is the left eigenvector of Γ with the eigenvalue zero. If the zero eigenvalue of Γ is degenerate, then the final value of the average particle numbers depend on the initial conditions. Also note that the second equation of Eq. (28) has one and only one solution for C'. The reason is that the real part of the eigenvalues of Λ are nonpositive. So, $v-u$ is not an eigenvalue of Λ, and hence the matrix $\Lambda - v + u$ is nonsingular.

If $u > v$, then the term $(u/v)^k$ diverges for the right end sites. So, we choose

$$N_k = B + C''\left(\frac{u}{v}\right)^{k-L}. \tag{29}$$

In the thermodynamics limit, one arrives at

$$\Lambda B = 0,$$

$$(\Gamma + v - u)C'' = -\Gamma B. \tag{30}$$

If $u > v$, it is the degeneracy of the zero eigenvalue of Λ, which determines the degeneracy of the stationary states of the particle numbers. Here too, the second equation of Eq. (30) determines C'' uniquely.

If $u < v$, the profile of average particle number is flat for the rightmost sites. As u exceeds v, it acquires a finite slope, proportional to $\ln(u/v)$. For the leftmost sites, this behavior is reversed. In fact, there is a discontinuous change of the behavior of the derivative of the stationary particle number at the end points. This is a static phase transition, and it is controlled only by the reaction rates and does not depend on the reaction rates at the boundaries. The case $q = 2$ has been already studied in [30]. One notes that the matrices Λ and Γ are

$$\Lambda = \begin{pmatrix} -a' & a \\ a' & -a \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} -b' & b \\ b' & -b \end{pmatrix} \tag{31}$$

in terms of the appropriate parameters of that references. Here, the zero eigenvalues of Λ and Γ are nondegenerate (unless the rates are zero).

Now let us consider the relaxation of the system towards its stationary state. The homogeneous part of Eq. (19) is

$$\dot{\delta N}_k^\alpha = h_{k\beta}^{l\alpha} \delta N_l^\beta, \tag{32}$$

where δN is the difference of N from the its stationary value, so that

$$s_\alpha \delta N_k^\alpha = 0. \tag{33}$$

To study the relaxation of the system, we investigate the eigenvalues of h. One finds

$$\begin{aligned} Ex_k^\alpha &= -(u+v)x_k^\alpha + ux_{k-1}^\alpha + vx_{k+1}^\alpha, \quad k \neq 1,L \\ Ex_1^\alpha &= \Lambda_\beta^\alpha x_1^\beta - vx_1^\alpha + vx_2^\alpha, \\ Ex_L^\alpha &= \Gamma_\beta^\alpha x_L^\beta - ux_L^\alpha + ux_{L-1}^\alpha, \end{aligned} \tag{34}$$

where the eigenvalue and eigenvector have been denoted by E and x, respectively. The solution to these equations is

$$x_k^\alpha = \mathcal{B}^\alpha z_1^k + \mathcal{C}^\alpha z_2^k, \tag{35}$$

where z_i's satisfy

$$E = -(u+v) + vz_i + \frac{u}{z_i}. \tag{36}$$

Performing the change of variable $z_i = \sqrt{u/v}Z_i$, the second and third equations of Eq. (34) take the form

$$\begin{aligned} \sqrt{\frac{u}{v}}E(\mathcal{B}Z + \mathcal{C}Z^{-1}) &= (\Lambda - v)\sqrt{\frac{u}{v}}(\mathcal{B}Z + \mathcal{C}Z^{-1}) \\ &\quad + v\left(\sqrt{\frac{u}{v}}\right)^2(\mathcal{B}Z^2 + \mathcal{C}Z^{-2}), \\ \sqrt{\frac{u}{v}}E(\mathcal{B}Z^L + \mathcal{C}Z^{-L}) &= (\Gamma - u)\left(\sqrt{\frac{u}{v}}\right)^L(\mathcal{B}Z^L + \mathcal{C}Z^{-L}) \\ &\quad + u\left(\sqrt{\frac{u}{v}}\right)^{L-1}(\mathcal{B}Z^{L-1} + \mathcal{C}Z^{-L+1}). \end{aligned} \tag{37}$$

Substituting $E = -(u+v) + \sqrt{uv}(Z + Z^{-1})$ in the above equations, one can write them in the matrix form

$$\begin{pmatrix} -(u+\Lambda)Z + \sqrt{uv} & -(u+\Lambda)Z^{-1} + \sqrt{uv} \\ -(v+\Gamma)Z^L + \sqrt{uv}Z^{L+1} & -(v+\Gamma)Z^{-L} + \sqrt{uv}Z^{-L-1} \end{pmatrix} \times \begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix} = 0. \tag{38}$$

To have nontrivial solutions for $\mathcal{B}$ and $\mathcal{C}$, the determinant of the matrix should be zero,

$$\det\begin{pmatrix} -(u+\Lambda)Z + \sqrt{uv} & -(u+\Lambda)Z^{-1} + \sqrt{uv} \\ -(v+\Gamma)Z^L + \sqrt{uv}Z^{L+1} & -(v+\Gamma)Z^{-L} + \sqrt{uv}Z^{-L-1} \end{pmatrix} = 0. \tag{39}$$

In the thermodynamic limit ($L \to \infty$), if all of the roots of Eq. (39) are phases, the largest real part of the eigenvalues of the Hamiltonian will be $-(u+v) + 2\sqrt{uv}$, from which the relaxation time of the system is

$$\tau_f = (u + v - 2\sqrt{uv})^{-1}. \tag{40}$$

We call this the *fast phase*.

If some of the roots of Eq. (40) are not phases, however, this may be not the case, there may be an eigenvalue of the Hamiltonian with a larger real part, which corresponds to a larger relaxation time. For $Z = re^{i\theta}$, one has

$$\mathrm{Re}(E) = -(u+v) + \sqrt{uv}\left(r + \frac{1}{r}\right)\cos\theta. \tag{41}$$

This leads to a larger relaxation time, provided

$$\left(r + \frac{1}{r}\right)\cos\theta > 2. \tag{42}$$

If there is such a solution for Z, the system is said to be in the *slow phase*.

So, if none of the roots of Eq. (39) satisfy Eq. (42), then the system is in the fast phase, and the relaxation time does not depend on the reaction matrices Γ and Λ. Otherwise, the relaxation time does depend on the reaction matrices (the slow phase). The transition between these two phases is the dynamical phase transition.

Now, let's seek the nonphase solutions of Eq. (39). If Z is a solution to Eq. (39), $1/Z$ is another solution to it. So it is sufficient to seek the solutions with $|Z| > 1$. In the thermodynamic limit, and for $|Z| > 1$, Eq. (39) is simplified to

$$\det[\sqrt{uv}Z - (v+\Gamma)]\det[\sqrt{uv}Z - (u+\Lambda)] = 0, \tag{43}$$

which consists of the characteristic equations for Γ and Λ. Denoting the eigenvalues of Γ and Λ by γ and λ, respectively, one has

$$Z = \frac{v+\gamma}{\sqrt{uv}}, \quad \text{or} \quad Z = \frac{u+\lambda}{\sqrt{uv}}. \tag{44}$$

As Γ and Λ are stochastic matrices, their eigenvalues have nonpositive real parts. For $u \geqslant v$, the real part of $(v+\gamma)/\sqrt{uv}$ is then not greater than 1. Hence, it cannot satisfy

Eq. (42). So, the only relevant equation for finding the system in the *slow phase* is

$$Z=\frac{u+\lambda}{\sqrt{uv}}. \tag{45}$$

A similar argument shows that for $v \geqslant u$, the first equation of Eq. (44) is relevant. One also concludes that for $u=v$, the system has no *slow phase*.

So, without loss of generality, let's take $u<v$. As Γ is a stochastic matrix, it has at least one, zero eigenvalue. However, if this eigenvalue is nondegenerate, the right eigenvector of Γ ($\mathcal{B}$) cannot satisfy

$$s_\alpha \mathcal{B}^\alpha = 0, \tag{46}$$

since s is the left eigenvector of Γ corresponding to the same eigenvalue. But from Eq. (33), it is seen that $\mathcal{B}$ should satisfy Eq. (46). So, from the eigenvalues of Γ, one should set aside one, zero eigenvalue, and consider only the other eigenvalues.

The system undergoes a dynamic phase transition at the point that for one of the Z's in Eq. (45) the criterion (42) is satisfied. At this point, the real and imaginary parts of Z satisfy

$$Y=\pm(X-1)\sqrt{\frac{X}{2-X}}, \quad X>1, \tag{47}$$

where

$$X=\mathrm{Re}(Z),$$

$$Y=\mathrm{Im}(Z). \tag{48}$$

One can translate this in terms of λ. The criterion for the *slow phase* is then seen to be

$$|\mathrm{Im}(\lambda)|<[\mathrm{Re}(\lambda)+u-\sqrt{uv}]\sqrt{\frac{\mathrm{Re}(\lambda)+u}{2\sqrt{uv}-\mathrm{Re}(\lambda)-u}},$$

$$\text{or} \quad \mathrm{Re}(\lambda)>2\sqrt{uv}-u. \tag{49}$$

A simple way to induce the phase transition is to multiply the matrix Λ by a parameter r. This means multiplying the rates of the reaction at the first site by r. As $\mathrm{Re}(\lambda)\leqslant 0$, one can see that for a large enough value of r, the value of $\mathrm{Re}(\lambda)+u-\sqrt{uv}$ will be negative [provided $\mathrm{Re}(\lambda)\neq 0$, that is, provided the zero eigenvalue of the matrix Λ is not degenerate]. So the system will be in the *fast phase*. It is also seen that as r tends to zero, either $2\sqrt{uv}-\mathrm{Re}(\lambda)-u$ becomes negative, or in the first inequality in Eq. (49) the right-hand side becomes greater than the left-hand side (which tends to zero). So, the system will be in the *slow phase*. Roughly speaking, increasing the reaction rates brings the system from the *slow phase* to the *fast phase*. A similar argument holds, of course, for the case $v>u$ and the eigenvalues of the matrix Γ.

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